THE ORDER OF A LINEARLY INVARIANT FAMILY IN $\mathbb{C}^n$

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Abstract. We study the (trace) order of the linearly invariant family in the ball $\mathbb{B}^n$ defined by $\|\mathcal{S}F\| \leq \alpha$, where $F : \mathbb{B}^n \to \mathbb{C}^n$ is locally biholomorphic and $\mathcal{S}F$ is the Schwarzian operator. By adapting Pommerenke's approach, we establish a characteristic equation for the extremal mapping that yields an upper bound for the order of the family in terms of α and the dimension n. Lower bounds for the order are established in similar terms by means of examples.

1. Introduction

The purpose of this paper is to obtain an upper bound for the trace order of a certain linearly invariant family of locally biholomorphic mappings defined in the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. The family is defined in terms of the Schwarzian derivative $\mathcal{S}F$, which inherits from the Bergman metric in $\mathbb{B}^n$ a natural norm $||\mathcal{S}F||$ that is invariant under the automorphism group [3]. Disregarding certain normalizations, the families $\mathcal{F}_\alpha$ considered in this paper are defined by the condition $||\mathcal{S}F|| \leq \alpha$. Linearly invariant families of holomorphic mappings were introduced in one complex variable by Pommerenke in two seminal papers that offered a systematic treatment of such families [11], [12]. He showed that relevant aspects of the family $\mathcal{F}$, such as growth and covering, are determined by its order $\sup_{f\in\mathcal{F}} |a_2(f)|$. If Sf is the usual Schwarzian derivative and $||Sf|| = \sup_{|z|<1}(1 - |z|^2)^2|Sf(z)|$, then the family of properly normalized locally univalent mappings in the disc $\mathbb{D}$ for which $||Sf|| \leq \alpha$ is linearly invariant. By means of a variational method, Pommerenke determined the sharp value $\sqrt{1 + \frac{1}{2}\alpha}$ for its order. In several variables, the concept of order of a linearly invariant family appears in the form of the (trace) order and the norm order, and both have implications on the growth of the family and on estimates on the jacobian [2]. In this work, we mimic the variational approach in several variables to estimate the order of $\mathcal{F}_\alpha$ in terms of α and the dimension n. Much like in the analysis found in [11], we are led to a characteristic equation involving derivatives of order up to three that must be satisfied by any mapping extremal for the trace order. The estimate on the trace order is then used to obtain

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a similar estimate for the norm order of the family $\mathcal{F}_\alpha$. In the final section, we consider examples in all dimensions to find lower bounds for the order.

2. Preliminaries

In [10] T.Oda generalizes the concept of Schwarzian derivative to the case of locally biholomorphic mappings in several variables. For such a mapping $F = (f_1, \ldots, f_n) : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ defined in a domain Ω in $\mathbb{C}^n$, he introduces a family of Schwarzian derivatives by

$$S_{ij}^k F = \sum_{l=1}^{n} \frac{\partial^2 f_l}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial f_l} - \frac{1}{n+1}\left(\delta_i^k \frac{\partial}{\partial z_j} + \delta_j^k \frac{\partial}{\partial z_i}\right) \log\, JF\,, \tag{2.1}$$

where $i, j, k = 1, 2, \ldots, n$, $JF = \det(DF)$ is the jacobian determinant of the diferential DF and δ_i^k are the Kronecker symbols. Two important aspects of the one dimensional Schwarzian are also present in this context. First,

$$S_{ij}^k F = 0 \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n \quad \text{iff} \quad F(z) = M(z)\,, \tag{2.2}$$

for some Möbius transformation

$$M(z) = \left(\frac{l_1(z)}{l_0(z)}, \ldots, \frac{l_n(z)}{l_0(z)}\right)\,,$$

where $l_i(z) = a_{i0} + a_{i1}z_1 + \cdots + a_{in}z_n$ with $\det(a_{ij}) \neq 0$. Next, under composition we have the chain rule

$$S_{ij}^k(G \circ F)(z) = S_{ij}^k F(z) + \sum_{l,m,r=1}^{n} S_{lm}^r G(w) \frac{\partial w_l}{\partial z_i} \frac{\partial w_m}{\partial z_j} \frac{\partial z_k}{\partial w_r}\,,\ w = F(z)\,. \tag{2.3}$$

Thus, if G is a Möbius transformation then $S_{ij}^k(G \circ F) = S_{ij}^k F$. The $S_{ij}^0 F$ coefficients are given by

$$S_{ij}^0 F(z) = (JF)^{\frac{1}{n+1}} \left(\frac{\partial^2}{\partial z_i \partial z_j}(JF)^{-\frac{1}{n+1}} - \sum_{k=1}^{n} \frac{\partial}{\partial z_k}(JF)^{-\frac{1}{n+1}} S_{ij}^k F(z)\right)\,.$$

One can find in the literature other equivalent formulations of the Schwarzian in several variables, which also come in the form of differential operators of orders two and three (see, *e.g.*, [6], [7], [9]). In order to recover a mapping from its Schwarzian derivatives we can consider the following overdetermined system of partial differential equations,

$$\frac{\partial^2 u}{\partial z_i \partial z_j} = \sum_{k=1}^{n} P_{ij}^k(z) \frac{\partial u}{\partial z_k} + P_{ij}^0(z) u\,, \quad i, j = 1, 2, \ldots, n\,, \tag{2.4}$$

where $z = (z_1, z_2, ..., z_n) \in \Omega$ and $P_{ij}^k(z)$ are holomorphic functions in Ω, for $k = 0, \ldots, n$ and $i, j = 1, \ldots, n$. The system (2.4) is called *completely integrable* if there are $n+1$ (maximum) linearly independent solutions. The system is said to be in *canonical form* (see [13]) if the coefficients satisfy

$$\sum_{j=1}^{n} P_{ij}^{j}(z) = 0\,, \quad i = 1, 2, \dots, n\,.$$

An important result established by Oda is that (2.4) is completely integrable and in canonical form if and only if $P_{ij}^k = S_{ij}^k F$ for a locally biholomorphic mapping $F = (f_1, \dots, f_n)$, where $f_i = u_i/u_0$ for $1 \le i \le n$ and $u_0, u_1, \dots, u_n$ is a set of linearly independent solutions of the system. It was also observed by Oda that $u_0 = (JF)^{-\frac{1}{n+1}}$ is always a solution of (2.4) with $P_{ij}^k = S_{ij}^k F$.

The individual components $S_{ij}^k F$ can be gathered to write an operator in the following form (see[3]).

Definition 2.1. For $k = 1, \dots, n$ let $\mathbb{S}^k F$ be the matrix

$$\mathbb{S}^k F = (S_{ij}^k F)\,, \quad i, j = 1, \dots, n\,.$$

Definition 2.2. We define the *Schwarzian derivative operator* as the bilinear mapping $\mathcal{S}F(z) : T_z\Omega \to T_z\Omega$ given by

$$\mathcal{S}F(z)(\vec{v}) = \left(\vec{v}^{\,t}\mathbb{S}^1 F(z)\vec{v}\,,\, \vec{v}^{\,t}\mathbb{S}^2 F(z)\vec{v}\,,\, \dots, \vec{v}^{\,t}\mathbb{S}^n F(z)\vec{v}\right)\,,$$

where $\vec{v} \in T_z\Omega$.

As an operator $\mathcal{S}F(z)$ inherits a norm from the metric in $T_z\Omega$:

$$\|\mathcal{S}F(z)\| = \sup_{\|\vec{v}\|=1} \|\mathcal{S}F(z)(\vec{v}\,)\|\,, \tag{2.5}$$

and finally, we let

$$||\mathcal{S}F|| = \sup_{z\in\Omega} ||\mathcal{S}F(z)||\,. \tag{2.6}$$

Our interest is to study certain classes of locally biholomorphic mappings F defined in the unit ball $\mathbb{B}^n$. The Bergman metric g on $\mathbb{B}^n$ is the hermitian product defined by

$$g_{ij}(z) = \frac{n+1}{(1-|z|^2)^2}\,\left[(1-|z|^2)\delta_{ij} + \bar{z}_i z_j\right]. \tag{2.7}$$

The automorphisms of $\mathbb{B}^n$ act as isometries of the Bergman metric, and are given by

$$\sigma(z) = \frac{Az+B}{Cz+D}\,,$$

where A is $n\times n$, B is $n\times 1$, C is $1\times n$ and D is 1×1 with

$$\begin{array}{rcl} A^t\overline{A}-C^t\overline{C} & = & \mathrm{Id}\,, \\ |D|^2-B^t\overline{B} & = & 1\,, \\ A^t\overline{B}-C^t\overline{D} & = & 0\,, \end{array}$$

(see, e.g., [5]).

By appealing to the chain rule (2.3), it was shown in [3] that

$$\|\mathcal{S}(F\circ\sigma)(z)\| = \|\mathcal{S}F(\sigma(z))\|\,,$$

from which

$$\|\mathcal{S}F\| = \|\mathcal{S}(F\circ\sigma)\|\,. \tag{2.8}$$

In this paper we will consider the family $\mathcal{F}_\alpha$ defined by

$$\mathcal{F}_\alpha = \{\, F:\mathbb{B}^n\to\mathbb{C}^n \mid F \text{ locally biholomorphic}\,,\ F(0)=0\,,\ DF(0)=\mathrm{Id}\,,\ \|\mathcal{S}F\|\le\alpha\,\}\,.$$

The family $\mathcal{F}_\alpha$ is linearly invariant and also compact [3]. We are interested in studying its (trace) order [2], given by

$$ord\,\mathcal{F}_\alpha = \sup_{F\in\mathcal{F}_\alpha}\ \sup_{|w|=1}\ \frac{1}{2}\left|\sum_{i,j=1}^{n}\frac{\partial^2 f_j}{\partial z_i\partial z_j}(0)w_i\right|\,. \tag{2.9}$$

Because the family is compact, the order is finite. An equivalent form of the order is given by

$$\mathcal{A}_\alpha = \sup_{f\in\mathcal{F}_\alpha}|\nabla(JF)(0)|\,,$$

which is shown in [3] to satisfy

$$\mathcal{A}_\alpha = 2ord\,\mathcal{F}_\alpha\,.$$

A second measure of the size of a linearly invariant family $\mathcal{F}$ is given by the norm order, defined by

$$||ord||\mathcal{F} = \sup_{f\in\mathcal{F}}\frac{1}{2}||D^2F(0)||\,,$$

where

$$F(z) = z+\frac{1}{2}D^2F(0)(z,z)+\cdots\,.$$

In general, $ord\,\mathcal{F}\le n||ord||\mathcal{F}$. For the family $\mathcal{F}_\alpha$ in particular, it was shown in [3] that

$$1+\frac{\sqrt{3}}{2}\,\alpha\ \le\ ||ord||\mathcal{F}_\alpha\ \le\ \frac{2}{n+1}\,ord\,\mathcal{F}_\alpha+\frac{\sqrt{n+1}}{2}\,\alpha\,. \tag{2.10}$$

3. Variations and Extremal Mappings

Let $F_0 \in \mathcal{F}_\alpha$ be a mapping for which $\mathcal{A}_\alpha$ is maximal, with $\nabla(JF_0)(0) = \Lambda = (\lambda_1, \ldots, \lambda_n)$. Let σ be an automorphism of $\mathbb{B}^n$ with $\sigma(0) = \zeta$, and consider the Koebe transform

$$G(z) = D\sigma(0)^{-1} DF_0(\zeta)^{-1} \left[F_0(\sigma(z)) - F_0(\zeta)\right].$$

The mapping $G \in \mathcal{F}_\alpha$ represents a variation of the extremal mapping F_0 when $|\zeta|$ is small. With this in mind, we need to compute $\nabla(JG)(0)$. We have that

$$DG(z) = D\sigma(0)^{-1} DF_0(\zeta)^{-1} DF_0(\sigma(z)) D\sigma(z),$$

hence

$$JG(z) = J\sigma(0)^{-1} JF_0(\zeta)^{-1} JF_0(\sigma(z)) J\sigma(z),$$

so that

$$\nabla(JG)(0) = \frac{\nabla(JG)}{JG}(0) = \frac{\nabla(JF_0)}{JF_0}(\zeta) D\sigma(0) + \frac{\nabla(J\sigma)}{J\sigma}(0). \tag{3.1}$$

In order to proceed with the analysis, we need the expansion of $\nabla(JG)(0)$ in powers of ζ.

Lemma 3.1. *Let*

$$B_{ij} = \sum_{k=1}^{n} S_{ij}^k F_0(0)\lambda_k \quad , \quad B_{ij}^0 = S_{ij}^0 F_0(0).$$

Then

$$\frac{\nabla(JF_0)}{JF_0}(\zeta) = \Lambda + A\cdot\zeta + O(|\zeta|^2) \ , \ |\zeta| \to 0,$$

where $A = (A_{ij})$ is the matrix given by

$$A_{ij} = B_{ij} - (n+1)B_{ij}^0 + \frac{\lambda_i\lambda_j}{n+1}. \tag{3.2}$$

Proof. Let $u_0 = (JF_0)^{-\frac{1}{n+1}}$ and $\phi(\zeta) = \frac{\nabla(JF_0)}{JF_0}(\zeta)$. Then $\phi(0) = \Lambda$ because $JF_0(0) = 1$. We have that $\phi = (\phi_1, \ldots, \phi_n)$, where

$$\phi_i(\zeta) = \partial_i \log(JF_0)(\zeta) = -(n+1)\partial_i(\log u_0)(\zeta) \ , \quad \partial_i = \partial/\partial z_i.$$

Since u_0 is a solution of (2.4) with $u_0(0) = 1$ and $\nabla u_0(0) = -\frac{1}{n+1}\nabla(JF_0)(0)$, we see that

$$\partial_j\phi_i(0) = -(n+1)\partial_j\left[\frac{\partial_i u_0}{u_0}\right](0) = -(n+1)\left[\partial_{ij}^2 u_0(0) - \partial_i u_0(0)\partial_j u_0(0)\right],$$

$$= -(n+1)\left[-\frac{B_{ij}}{n+1} + B_{ij}^0 - \frac{\lambda_i\lambda_j}{(n+1)^2}\right],$$

which gives that the differential $D\phi(0)$ is given by the matrix $A = (A_{ij})$. This proves the lemma. $\square$

Lemma 3.2. *With the notation as before, one can choose σ so that*

$$D\sigma(0) = \mathrm{Id} + O(|\zeta|^2)\,,$$
$$\frac{\nabla(J\sigma)}{J\sigma}(0) = -(n+1)\overline{\zeta}\,.$$

Proof. Assume first that $\sigma(0) = \zeta = (\zeta_1, 0, \ldots, 0)$. Then we may take

$$\sigma(z) = \left(\frac{z_1+\zeta_1}{1+\overline{\zeta}_1 z_1}, \frac{\sqrt{1-|\zeta_1|^2}z_2}{1+\overline{\zeta}_1 z_1}, \ldots, \frac{\sqrt{1-|\zeta_1|^2}z_n}{1+\overline{\zeta}_1 z_1}\right)\,,$$

and one finds that

$$J\sigma(z) = \frac{(1-|\zeta_1|^2)^{\frac{n+1}{2}}}{(1+\overline{\zeta}_1 z_1)^{n+1}}\,,$$

together with

$$D\sigma(0) = \begin{pmatrix} (1-|\zeta_1|^2) & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{1-|\zeta_1|^2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{1-|\zeta_1|^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{1-|\zeta_1|^2} \end{pmatrix}\,,$$

from which the lemma follows for ζ of the form $(\zeta_1, 0, \ldots, 0)$. The general case obtains after considering a rotation of the ball. $\square$

In light of Lemmas 3.1 and 3.2, we can rewrite equation (3.1) as

$$\nabla(JG)(0) = \Lambda + A\cdot\zeta - (n+1)\overline{\zeta} + O(|\zeta|^2)\,. \tag{3.3}$$

Theorem 3.3. *Let $F_0 \in \mathcal{F}_\alpha$ be extremal for the order, with $\nabla(JF_0)(0) = \Lambda$. Then*

$$A\cdot\overline{\Lambda} = (n+1)\overline{\Lambda}\,. \tag{3.4}$$

Proof. The proof is based on the observation that, in reference to equation (3.3), we must have

$$|\nabla(JG)(0)| \leq |\Lambda| \;\;,\;\; |\zeta| \to 0\,.$$

Let $\langle v, w\rangle = v_1\overline{w_1} + \cdots + v_n\overline{w_n}$. Then

$$\begin{aligned}|\nabla(JG)(0)|^2 &= |\Lambda|^2 + \mathrm{Re}\langle\Lambda, A\cdot\zeta\rangle - 2(n+1)\mathrm{Re}\langle\Lambda,\zeta\rangle + O(|\zeta|^2) \\ &= |\Lambda|^2 + 2\mathrm{Re}\langle\overline{A^t}\cdot\Lambda - (n+1)\Lambda, \zeta\rangle + O(|\zeta|^2)\,.\end{aligned}$$

Since $\zeta = (\zeta_1, \ldots, \zeta_n)$ can be chosen small but otherwise arbitrary, we conclude that

$$\overline{A^t}\cdot\Lambda - (n+1)\Lambda = 0\,,$$

which proves the theorem because $A^t = A$. □

In order to facilitate the use of equation (3.4) to estimate the order of $\mathcal{F}_\alpha$, we use linear invariance to assume that $\Lambda = (\lambda, 0, \ldots, 0)$ with $\lambda > 0$. This normalization has a decoupling effect on (3.4), with the matrix A now given by

$$A_{ij} = S^1_{ij}\lambda - (n+1)S^0_{ij} + \frac{\delta^1_i \delta^1_j}{n+1}\,\lambda^2\,, \tag{3.5}$$

where $S^k_{ij} = S^k_{ij}F_0(0)$. By equating the first components of (3.4) we obtain

$$\lambda^2 + (n+1)S^1_{11}\lambda - (n+1)^2 S^0_{11} - (n+1)^2 = 0\,, \tag{3.6}$$

while the remaining components give

$$S^1_{1j}\lambda - (n+1)S^0_{1j} = 0 \;\;,\;\; j = 2, \ldots, n\,. \tag{3.7}$$

We are now in position to estimate the order of the family $\mathcal{F}_\alpha$.

Theorem 3.4. *The order of $\mathcal{F}_\alpha$ satisfies*

$$ord\,\mathcal{F}_\alpha \le \frac{1}{2}(n+1)\left[\frac{1}{2}\sqrt{n+1}\alpha + \sqrt{1 + \frac{1}{4}(n+1)\alpha^2 + C(n,\alpha)}\,\right]\,, \tag{3.8}$$

where

$$C(n,\alpha) \le 6n^2\alpha^2 + 16\sqrt{n}\alpha\,.$$

Proof. From (3.7), we see that

$$\left(\lambda + \frac{1}{2}(n+1)S^1_{11}\right)^2 = (n+1)^2\left(1 + \frac{1}{4}(S^1_{11})^2 + S^0_{11}\right)\,.$$

In [3], the following bounds were established for the quantities S^1_{11}, S^0_{11}:

$$|S^1_{11}| \le \sqrt{n+1}\alpha \quad , \quad |S^0_{11}| \le C(n,\alpha)\,,$$

where

$$C(n,\alpha) = \left(4n^2 + 2n - 2 + \frac{n+1}{n-1}\right)\alpha^2 + \left(4\sqrt{n+1} + 8\frac{\sqrt{n+1}}{n-1}\right)\alpha\,.$$

The inequality (3.8) follows at once from the estimates on S^1_{11}, S^0_{11}. Finally, it is not difficult to see that

$$C(n,\alpha) \le 6n^2\alpha^2 + 16\sqrt{n}\alpha\,.$$

□

The following corollary is obtained at once from (2.10).

Corollary 3.5. *For the family $\mathcal{F}_\alpha$ we have*

$$||ord||\mathcal{F}_\alpha \;\le\; (n+1)\alpha + \sqrt{1 + \frac{1}{4}(n+1)\alpha^2 + C(n,\alpha)}\,.$$

4. Some Examples

In this section we construct examples in all dimensions to establish lower bounds for the $\mathcal{A}_\alpha$. When $n=1$ the task of constructing good examples is much simpler due to the nature of the differential equation associated with the Schwarzian. In fact, in this way one can show the sharpness of the estimate obtained from the variational method. In several variables, the complexity of the Schwarzian system is considerably higher. To simplify matters, we will consider the case when all S_{ij}^kF are constants. Two lemmas will be important in this process. The first lemma was established in [1], but we include the proof for the convenience of the reader.

Lemma 4.1. *Let u be a solution of a completely integrable system of the form (2.4) with $P_{ij}^k = S_{ij}^kF$ for some locally biholomorphic mapping F defined in Ω. Then there exists a Möbius transformation T such that $u=(JG)^{-\frac{1}{n+1}}$ for $G=T\circ F$.*

Proof. We write $F=(u_1/u_0,\ldots,u_n/u_0)$ for $n+1$ linearly independent solutions $u_0,u_1,\ldots,u_n$ of (2.4) with $u_0=(JF)^{-\frac{1}{n+1}}$. Then $u=b_0u_0+b_1u_1+\cdots+b_nu_n$ for some unique constants b_i. A simple calculation shows that $(JT)^{-\frac{1}{n+1}}=a_0+a_1w_1+\cdots+a_nw_n=l_0(w)$ whenever T is a Möbius of the form $(w_1/l_0(w),\ldots,w_n/l_0(w))$. Then

$$\begin{aligned}(J(T\circ F))^{-\frac{1}{n+1}} &= (JT(F))^{-\frac{1}{n+1}}(JF)^{-\frac{1}{n+1}}\\ &= (a_0+a_1f_1+\cdots+a_nf_n)u_0\\ &= a_0u_0+a_1u_1+\cdots+a_nu_n\,,\end{aligned}$$

hence it suffices to choose T with the property that $(JT)^{-\frac{1}{n+1}}=b_0+b_1z_1+\cdots+b_nz_n$. Note that the zero set of u is given by the hypersurface $a_0+a_1f_1+\cdots+a_nf_n=0$, that is, exactly the set where G becomes singular. $\square$

It follows from the lemma that if $u\neq 0$ is a solution of (2.4), then the mapping G will be regular in Ω. Thus, for the purpose of finding lower bounds for the order, it will suffice to exhibit solutions u that are non-vanishing in $\mathbb{B}^n$, and which have $u(0)=1$ together with $|\nabla u(0)|$ large in comparison to the norm $||SF||=||SG||$. The second lemma, of general interest, involves estimating $||SF||$ when all S_{ij}^kF are constant.

Lemma 4.2. *Let F be a locally biholomorphic mapping defined in $\mathbb{B}^n$ for which S_{ij}^k are constant, for all i,j,k. Then $||\mathcal{S}F||=||\mathcal{S}F(0)||$.*

Proof. The Bergman metric applied to vector $\vec{v}$ can be expressed by

$$\|\vec{v}\|^2=\frac{n+1}{(1-|z|^2)^2}\left[(1-|z|^2)|\vec{v}|^2+|z_1v_1+\cdots z_nv_n|^2\right]\,,$$

where $z=(z_1,\dots,z_n)$ and $\vec{v}=(v_1,\dots,v_n)$. Using the Cauchy-Schwartz inequality, we have that

$$\|\vec{v}\|^2 \le \frac{n+1}{(1-|z|^2)^2}|\vec{v}|^2\,.$$

Suppose that F has $S_{ij}^k F$ constants. Thus

$$\|SF(z)(\vec{v},\vec{v})\| \le \frac{|SF(z)(\vec{v},\vec{v})|\sqrt{n+1}}{1-|z|^2}\,.$$

But $\|\vec{v}\|=1$, therefore $|\vec{v}|^2 \le (1-|z|^2)/(n+1)$ and

$$\|SF(z)(\vec{v},\vec{v})\| = \left\|SF(z)\left(\frac{\vec{v}}{\|\vec{v}\|_0},\frac{\vec{v}}{\|\vec{v}\|_0}\right)\right\|_0 \frac{\|\vec{v}\|_0^2}{1-|z|^2} \le \left\|SF(z)\left(\frac{\vec{v}}{\|\vec{v}\|_0},\frac{\vec{v}}{\|\vec{v}\|_0}\right)\right\|_0\,,$$

where $\|\cdot\|_0$ means the Bergman metric at the origin. Taking supremum over $\|\vec{v}\|=1$ we have that

$$\|SF(z)\| \le \|SF(z)\|_0 = \|SF(0)\|\,.$$

□

Theorem 4.3. *The order of the family $\mathcal{F}_\alpha$ satisfies*

$$ord\,\mathcal{F}_\alpha \ge \frac{9}{4}\alpha\ \ ,\ \ n=2 \tag{4.1}$$

$$ord\,\mathcal{F}_\alpha \ge \frac{1}{2}\frac{(n+1)^{\frac{3}{2}}}{n-1}\alpha\ \ ,\ \ n>2\,. \tag{4.2}$$

Proof. We will exhibit a non-vanishing solution u of (2.4) that satisfies $u(0)=1$ and $\nabla u(0)$ large. In the system we let $S_{11}^1 F = -\sqrt{n+1}\alpha$ and $S_{1j}^1 F = 0$ for $j=2,\dots,n$. The integrability conditions allow us to set

$$S_{1k}^k F = \frac{\sqrt{n+1}}{n-1}\alpha\ \ ,\ \ k=2,\dots,n\,.$$

With $v=(v_1,\dots,v_n)$ we have that

$$\mathcal{S}F(0)(\vec{v}) = \left(-\sqrt{n+1}\alpha\, v_1^2\,,\, 2\frac{\sqrt{n+1}}{n-1}\alpha\, v_1v_2\,,\,\dots\,,\, 2\frac{\sqrt{n+1}}{n-1}\alpha\, v_1v_n\right)\,,$$

and so

$$\|\mathcal{S}F(0)(\vec{v})\|^2 = (n+1)^2\alpha^2\left[|v_1|^4 + \frac{4}{(n-1)^2}|v_1|^2(|v_2|^2+\cdots+|v_n|^2)\right]\,.$$

But $\|\vec{v}\|=1$, hence

$$\|\mathcal{S}F(0)(\vec{v})\|^2 = (n+1)^2\alpha^2|v_1|^2\left[|v_1|^2 + \frac{4}{(n-1)^2}\left(\frac{1}{n+1}-|v_1|^2\right)\right]\,.$$

In order to determine the maximum value of $\|\mathcal{S}F(0)(\vec{v})\|$ we consider the function

$$h(x) = x\left[\frac{4}{(n-1)^2(n+1)} + \left(1 - \frac{4}{(n-1)^2}\right)x\right]$$

in the interval $x \in [0, 1/(n+1)]$. The analysis of the maximum value of $h(x)$ must take into consideration that the term $1 - 4(n-1)^{-2}$ is negative for $n = 2$ and positive for $n > 2$.

For $n = 2$, the function $h(x) = (4/3)x - 3x^2$, which attains its maximum at $x = 2/9$ with $h(2/9) = 4/27$. Therefore

$$||\mathcal{S}F(0)|| = \frac{2}{\sqrt{3}}\alpha = \beta\,.$$

By Lemma 4.2, $||\mathcal{S}F|| = (2/\sqrt{3})\alpha$.

For $n > 2$, the function $h(x)$ attains its maximum at $x = 1/\sqrt{n+1}$, with corresponding maximal value

$$\frac{1}{n+1}\left(\frac{4}{(n-1)^2(n+1)} + \left[1 - \frac{4}{(n-1)^2}\right]\frac{1}{n+1}\right) = \frac{1}{(n+1)^2}\,.$$

Therefore $\|\mathcal{S}F(0)\|^2 = \alpha^2$, or equivalently,

$$\|\mathcal{S}F(0)\| = \alpha\,.$$

By Lemma 4.2, then $||\mathcal{S}F|| = \alpha$.

On the other hand, the system reads

$$\begin{aligned} u_{11} &= -\sqrt{n+1}\alpha\, u_1 + \frac{n\sqrt{n+1}}{n-1}\alpha^2 u \\ u_{1j} &= \frac{\sqrt{n+1}}{n-1}\alpha\, u_j\,, \quad j > 1 \\ u_{ij} &= 0\,, \quad i, j > 1\,. \end{aligned}$$

Consider the solution with $u(0) = 1$ and $\nabla u(0) = (\lambda, 0, \ldots, 0)$. Since $u_{ij} = 0$ for all $i, j > 1$, then

$$u(z) = a_n(z_1)z_n + a_{n-1}(z_1)z_{n-1} + \ldots a_2(z_1)z_2 + a_1(z_1)\,.$$

Now, $\partial u/\partial z_j = a_j(z_1)$ for $j > 1$. From the second equation we have that

$$a_j'(z_1) = \frac{\sqrt{n+1}}{n-1}\alpha a_j(z_1)\,,$$

which implies that for constants c_j

$$a_j(z_1) = c_j e^{\frac{\sqrt{n+1}}{n-1}\alpha z_1}\,, \quad j > 1\,.$$

Since $\nabla u(0) = (\lambda, 0, \ldots, 0)$ it follows that $a_j(0) = 0$, therefore $c_j = 0$ an $a_j \equiv 0$ for $j > 1$. We conclude that

$$u(z) = a_1(z_1)\,.$$

With the notation $a_1(z_1) = a(z)$ we use the first equation of the system

$$a''(z) = -\sqrt{n+1}\alpha\, a'(z) + \frac{n\sqrt{n+1}}{n-1}\alpha^2 a(z)\,,$$

to conclude that

$$u(z) = a(z_1) = C_1 e^{\frac{\sqrt{n+1}}{n-1}\alpha z_1} + C_2 e^{-\frac{n\sqrt{n+1}}{n-1}\alpha z_1}\,.$$

Considering that $a(0) = 1$ and $a'(0) = \lambda$, and putting $\lambda = \sqrt{n+1}/(n-1)\alpha$, then

$$u(z) = e^{\frac{\sqrt{n+1}}{n-1}\alpha z_1}\,,$$

which does not vanish in $\mathbb{B}^n$.

Suppose $n = 2$. By Lemma 4.1 and the preceding analysis, there exists $G \in \mathcal{F}_\beta$ with $JG = u^{-3}$. Hence $\nabla(JG)(0) = -3\nabla u(0) = -3\sqrt{3}\alpha = \frac{9}{2}\beta$, which shows that $\mathcal{A}_\beta \geq \frac{9}{2}\beta$, and thus proving (4.1).

By the same token, for $n > 2$ there exists $G \in \mathcal{F}_\alpha$ with $JG = u^{-(n+1)}$. This mapping has $\nabla(JG)(0) = -(n+1)\left[\sqrt{n+1}/(n-1)\right]\alpha$, which proves (4.2). $\square$

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References

[1] Chuaqui, M. and Hernández, R., Ahlfors-Weill extensions in several complex variables, to appear in *J. Reine Angew. Math.*

[2] Graham, I. and Kohr, G., Geometric function theory in one and higher dimensions, *Pure and Applied Math.* **255**, Marcel Dekker (2003).

[3] Hernández, R., Schwarzian derivatives and a linearly invariant family in $\mathbb{C}^n$, *Pacific J. Math.* **228** (2006), 201-218.

[4] Hernández, R., Schwarzian derivatives and some criteria for univalence in $\mathbb{C}^n$, *Complex Variables* **52** (2007), 397-410.

[5] Krantz, S., Function Theory of Several Complex Variables, *Pure and Applied Math.*, Wiley (1982).

[6] Molzon, R. and Mortensen, K.P., Differential operators associated with holomorphic mappings, *Annals of Global Analysis and Geometry* **12** (1994), 291-304.

[7] Molzon, R. and Mortensen, K. P., The Schwarzian derivative for maps between manifolds with complex projective connections. *Trans. Amer. Math. Soc.* **348** (1996), 3015-3036.

[8] Molzon, R. and Mortensen, K.P., Univalence of Holomorphic Mappings, *Pacific J. Math.* **180** (1997), 125-133.

[9] Molzon, R. and Tamanoi H., Generalized Schwarzians in several variables and Möbius invariant differential operators, *Forum Math.* **14** (2002), 165-188.

[10] Oda, T., On Schwarzian Derivatives in several variables (in Japanese), *Kokyuroku of R.I.M., Kyoto Univ.* **226** (1974).

[11] Pommerenke, Ch., Linear-invariante Familien analytischer Funktionen I, *Math. Ann.* **155** (1964), 108-154.
[12] Pommerenke, Ch., Linear-invariante Familien analytischer Funktionen II, *Math. Ann.* **156** (1964), 226-262.
[13] Yoshida, M., Canonical forms of some systems of linear partial differential equations, *Proc. Japan Acad.* **52** (1976), 473-476.
[14] Yoshida, M., Orbifold-Uniformizing Differential Equations, *Math. Ann.* **267** (1984), 125-142.

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